Annexure I

CLASS XI

CHAPTER 3

3.6. Proofs and Simple Applications of sine and cosine formulae

Let ABC be a triangle. By angle A we mean the angle between the sides AB and AC which lies between 0° and 180°. The angles B and C are similarly defined. The sides AB, BC and CA opposite to the vertices C, A and B will be denoted by c, a and b, respectively (see Fig. 3.15).

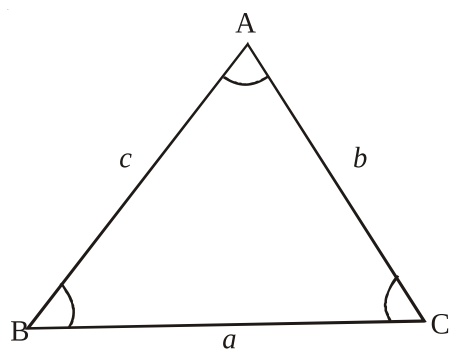


Fig. 3.15

Theorem 1 (sine formula) In any triangle, sides are proportional to the sines of the opposite angles. That is, in a triangle ABC

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Proof Let ABC be either of the triangles as shown in Fig. 3.16 (i) and (ii).

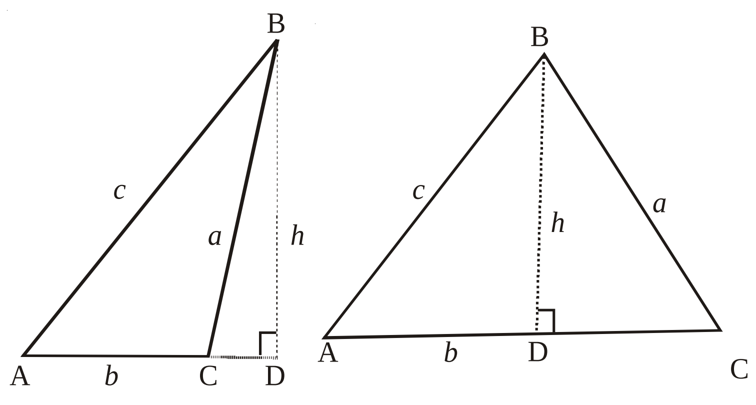


Fig. 3.16

The altitude h is drawn from the vertex B to meet the side AC in point D [in (i) AC is produced to meet the altitude in D]. From the right angled triangle ABD in Fig. 3.16(i), we have

$$\sin A = \frac{h}{c}, \text{ i.e., } h = c \sin A \tag{1}$$

and $$\sin (180° - C) = \frac{h}{a} \Rightarrow h = a \sin C \tag{2}$$

From (1) and (2), we get

$$c \sin A = a \sin C, \text{ i.e., } \frac{\sin A}{a} = \frac{\sin C}{c} \tag{3}$$

Similarly, we can prove that

$$\frac{\sin A}{a} = \frac{\sin B}{b} \qquad (4)$$

From (3) and (4), we get

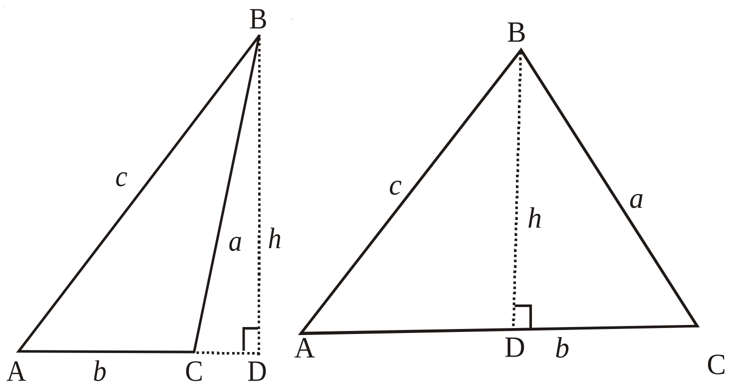
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

For triangle ABC in Fig. 3.16 (ii), equations (3) and (4) follow similarly.

Theorem 2 (Cosine formulae) Let A, B and C be angles of a triangle and a, b and c be lengths of sides opposite to angles A, B and C, respectively, then

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = c^2 + a^2 - 2ca\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Proof Let ABC be triangle as given in Fig. 3.17 (i) and (ii)



(i) (ii)

Fig. 3.17

Referring to Fig.3.17 (ii), we have

$$BC^2 = BD^2 + DC^2 = BD^2 + (AC - AD)^2$$
$$= BD^2 + AD^2 + AC^2 - 2AC.AD$$
$$= AB^2 + AC^2 - 2AC\,AB\cos A$$

or $\quad a^2 = b^2 + c^2 - 2bc \cos A$

Similarly, we can obtain

$$b^2 = c^2 + a^2 - 2ca\cos B$$

and $\quad c^2 = a^2 + b^2 - 2ab\cos C$

Same equations can be obtained for Fig. 3.17 (i), where C is obtuse.

A convenient form of the cosine formulae, when angles are to be found are as follows:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Example 25 In triangle ABC, prove that

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$$
$$\tan\frac{C-A}{2}=\frac{c-a}{c+a}\cot\frac{B}{2}$$
$$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$$

Proof By sine formula, we have

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k(say).$$

Therefore, $$\frac{b-c}{b+c}=\frac{k(\sin B-\sin C)}{k(\sin B+\sin C)}$$

$$=\frac{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}$$

$$=\cot\frac{(B+C)}{2}\tan\frac{(B-C)}{2}$$

$$=\cot\left(\frac{\pi}{2}-\frac{A}{2}\right)\tan\left(\frac{B-C}{2}\right)$$

$$=\frac{\tan\frac{B-C}{2}}{\cot\frac{A}{2}}$$

Therefore, $$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$$

Similarly, we can prove other results. These results are well known as Napier's Analogies.

Example 26 In any triangle ABC, prove that

$a\sin(B-C)+b\sin(C-A)+C\sin(A-B)=0$

Solution Consider

$a\sin(B-C)=a[\sin B\cos C-\cos B\sin C]$ (1)

Now $$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}=k(\text{say})$$

Therefore, $\sin A=ak$, $\sin B=bk$, $\sin C=ck$

Substituting the values of sinB and sinC in (1) and using cosine formula, we get

$$a\sin(B-C)=a\left[bk\left(\frac{a^2+b^2-c^2}{2ab}\right)-ck\left(\frac{c^2+a^2-b^2}{2ac}\right)\right]$$

$$= \frac{k}{2}\left(a^2 + b^2 - c^2 - c^2 - a^2 + b^2\right)$$

$$= k(b^2 - c^2)$$

Similarly, $b \sin (C - A) = k (c^2 - a^2)$

and $\quad c\sin (A - B) = k (a^2 - b^2)$

Hence L.H.S $= k (b^2 - c^2 + c^2 - a^2 + a^2 - b^2)$

$= 0 =$ R.H.S.

Example 27 The angle of elevation of the top point P of the vertical tower PQ of height h from a point A is 45° and from a point B, the angle of elevation is 60°, where B is a point at a distance d from the point A measured along the line AB which makes an angle 30° with AQ. Prove that $d = h(\sqrt{3} - 1)$

Proof From the Fig. 3.18, we have ∠PAQ = 45°, ∠BAQ = 30°, ∠ PBH = 60°

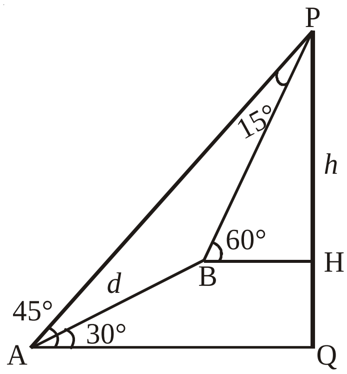


Fig. 3.18

Clearly $\quad \angle APQ = 45°, \angle BPH = 30°,$ giving $\angle APB = 15°$

Again $\quad \angle PAB = 15° \Rightarrow \angle ABP = 150°$

From triangle APQ, we have $AP^2 = h^2 + h^2 = 2h^2$ (Why ?)

or $\quad AP = \sqrt{2}h$

Applying sine formula in Δ ABP, we get

$$\frac{AB}{\sin 15°} = \frac{AP}{\sin 150°} \Rightarrow \frac{d}{\sin 15°} = \frac{\sqrt{2}h}{\sin 150°}$$

i.e., $\quad d = \dfrac{\sqrt{2}h \sin 15°}{\sin 30°}$

$= h(\sqrt{3} - 1)$ (why?)

Example 28 A lamp post is situated at the middle point M of the side AC of a triangular plot ABC with BC = 7m, CA = 8m and AB = 9 m. Lamp post subtends an angle 15° at the point B. Determine the height of the lamp post.

Solution From the Fig. 3.19, we have AB = 9 = c, BC = 7 = a and AC = 8 = b.

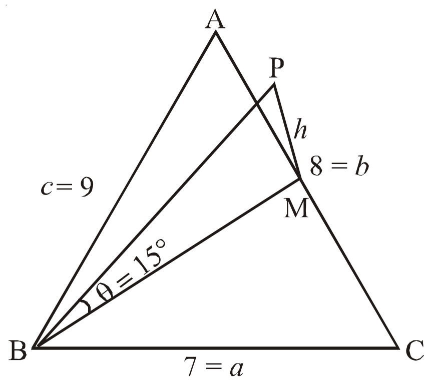


Fig.3.19

M is the mid point of the side AC at which lamp post MP of height h (say) is located. Again, it is given that lamp post subtends an angle θ (say) at B which is 15°.

Applying cosine formula in ΔABC, we have

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{49 + 64 - 81}{2 \times 7 \times 8} = \frac{2}{7} \qquad (1)$$

Similarly using cosine formula in ΔBMC, we get

$$BM^2 = BC^2 + CM^2 - 2\ BC \times CM \cos C.$$

Here $CM = \frac{1}{2} CA = 4$, since M is the mid point of AC.

Therefore, using (1), we get

$$BM^2 = 49 + 16 - 2 \times 7 \times 4 \times \frac{2}{7}$$

$$= 49$$

or $\quad BM = 7$

Thus, from ΔBMP right angled at M, we have

$$\tan\theta = \frac{PM}{BM} = \frac{h}{7}$$

or $\quad \frac{h}{7} = \tan(15°) = 2 - \sqrt{3} \qquad$ (why ?)

or $\quad h = 7(2 - \sqrt{3})\ \mathrm{m}$.

EXERCISE 3.5

In any triangle ABC, if $a = 18$, $b = 24$, $c = 30$, find

1. cosA, cosB, cosC (Ans. $\frac{4}{5}, \frac{3}{5}, 0$) **2.** sinA, sinB, sinC (Ans. $\frac{3}{5}, \frac{4}{5}, 1$)

For any triangle ABC, prove that

3. $\dfrac{a+b}{c} = \dfrac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}}$ **4.** $\dfrac{a-b}{c} = \dfrac{\sin\left(\frac{A-B}{2}\right)}{\cos\frac{C}{2}}$

5. $\sin\frac{B-C}{2}=\frac{b-c}{a}\cos\frac{A}{2}$

6. $a(b\cos C - c\cos B) = b^2 - c^2$

7. $a(\cos C - \cos B) = 2(b-c)\cos^2\frac{A}{2}$

8. $\frac{\sin(B-C)}{\sin(B+C)}=\frac{b^2-c^2}{a^2}$

9. $(b+c)\cos\frac{B+C}{2}=a\cos\frac{B-C}{2}$

10. $a\cos A+b\cos B+c\cos C=2a\sin B\sin C$

11. $\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}=\frac{a^2+b^2+c^2}{2abc}$

12. $(b^2-c^2)\cot A+(c^2-a^2)\cot B+(a^2-b^2)\cot C=0$

13. $\frac{b^2-c^2}{a^2}\sin 2A+\frac{c^2-a^2}{b^2}\sin 2B+\frac{a^2-b^2}{c^2}\sin 2C=0$

14. A tree stands vertically on a hill side which makes an angle of 15° with the horizontal. From a point on the ground 35m down the hill from the base of the tree, the angle of elevation of the top of the tree is 60°. Find the height of the tree. (Ans. $35\sqrt{2}m$)

15. Two ships leave a port at the same time. One goes 24 km per hour in the direction N45°E and other travels 32 km per hour in the direction S75°E. Find the distance between the ships at the end of 3 hours. (Ans. 86.4 km (approx.))

16. Two trees, A and B are on the same side of a river. From a point C in the river the distance of the trees A and B is 250m and 300m, respectively. If the angle C is 45°, find the distance between the trees (use $\sqrt{2}$ =1.44). (Ans. 215.5 m)

CHAPTER 5

5.7. Square-root of a Complex Number

We have discussed solving of quadratic equations involving complex roots on page 108-109 of textbook. Here we explain the particular procedure for finding square root of a complex number expressed in the standard form. We illustrate the same by an example.

Example 12 Find the square root of $-7-24i$

Solution Let $x+iy=\sqrt{-7-24i}$

Then $(x+iy)^2=-7-24i$

or $x^2-y^2+2xyi=-7-24i$

Equating real and imaginary parts, we have

$$x^2-y^2=-7 \qquad (1)$$

$$2xy=-24$$

We know the identity

$$\left(x^2+y^2\right)^2=\left(x^2-y^2\right)^2+(2xy)^2$$

$= 49 + 576$

$= 625$

Thus, $x^2+y^2=25$ (2)

From (1) and (2), $x^2 = 9$ and $y^2 = 16$

or $\qquad x = \pm 3$ and $y = \pm 4$

Since the product xy is negative, we have

$x = 3, y = -4$ or, $x = -3, y = 4$

Thus, the square roots of $-7 - 24i$ are $3 - 4i$ and $-3 + 4i$

EXERCISE 5.4

Find the square roots of the following:

1. $-15 - 8i$ (Ans. $1 - 4i, -1 + 4i$) **2.** $-8 - 6i$ (Ans. $1 - 3i, -1 + 3i$)

3. $1 - i$ (Ans. $\left(\pm\sqrt{\dfrac{\sqrt{2}+1}{2}}\ \mu \sqrt{\dfrac{\sqrt{2}-1}{2}}i\right)$) **4.** $-i$ (Ans. $\left(\pm\dfrac{1}{\sqrt{2}}\ \mathbf{m}\dfrac{1}{\sqrt{2}}i\right)$)

5. i (Ans. $\left(\pm\dfrac{1}{\sqrt{2}}\pm\dfrac{1}{\sqrt{2}}i\right)$) **6.** $1 + i$ (Ans. $\left(\pm\sqrt{\dfrac{\sqrt{2}+1}{2}}\pm\sqrt{\dfrac{\sqrt{2}-1}{2}}i\right)$)

CHAPTER 9

9.7. Infinite G.P. and its Sum

G. P. of the form $a, ar, ar^2, ar^3, ...$ is called infinite G. P. Now, to find the formula for finding sum to infinity of a G. P., we begin with an example.

Let us consider the G. P.,

$$1, \frac{2}{3}, \frac{4}{9}, ...$$

Here $\quad a = 1,\ r = \dfrac{2}{3}$. We have

$$S_n = \frac{1 - \left(\frac{2}{3}\right)^n}{1 - \frac{2}{3}} = 3\left[1 - \left(\frac{2}{3}\right)^n\right]$$

Let us study the behaviour of $\left(\dfrac{2}{3}\right)^n$ as n becomes larger and larger:

n	1	5	10	20
$\left(\frac{2}{3}\right)^n$	0.6667	0.1316872428	0.01734152992	0.00030072866

We observer that as n becomes larger and larger, $\left(\dfrac{2}{3}\right)^n$ becomes closer and closer to zero. Mathematically, we say that as n becomes sufficiently large, $\left(\dfrac{2}{3}\right)^n$ becomes sufficiently small. In other words as

$n \to \infty, \left(\frac{2}{3}\right)^n \to 0$. Consequently, we find that the sum of infinitely many terms is given by $S_\infty = 3$.

Now, for a geometric progression, $a, ar, ar^2, ...,$ if numerical value of common ratio r is less than 1, then

$$S_n = \frac{a(1-r^n)}{(1-r)} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

In this case as $n \to \infty, r^n \to 0$ since $|r| < 1$. Therefore

$$S_n \to \frac{a}{1-r}$$

Symbolically sum to infinity is denoted by S_∞ or S.

Thus, we have $S = \frac{a}{1-r}$.

For examples

(i) $$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ... = \frac{1}{1 - \frac{1}{2}} = 2.$$

(ii) $$1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + ... = \frac{1}{1 - \left(\frac{-1}{2}\right)} = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}$$

EXERCISE 9.4

Find the sum to infinity in each of the following Geometric Progression.

1. $1, \frac{1}{3}, \frac{1}{9}, ...$ (Ans. 1.5)

2. 6, 1.2, .24, ... (Ans. 7.5)

3. $5, \frac{20}{7}, \frac{80}{49}, ...$ (Ans. $\frac{35}{3}$)

4. $\frac{-3}{4}, \frac{3}{16}, \frac{-3}{64}, ...$ (Ans. $\frac{-3}{5}$)

5. Prove that $3^{\frac{1}{2}} \times 3^{\frac{1}{4}} \times 3^{\frac{1}{8}} ... = 3$

6. Let $x = 1 + a + a^2 + ...$ and $y = 1 + b + b^2 + ...$, where $|a| < 1$ and $|b| < 1$. Prove that

$$1 + ab + a^2b^2 + ... = \frac{xy}{x + y - 1}$$

CHAPTER 10

10.6 Equation of family of lines passing through the point of intersection of two lines

Let the two intersecting lines l_1 and l_2 be given by

$$A_1x + B_1y + C_1 = 0 \quad (1)$$

and

$$A_2x + B_2y + C_2 = 0 \quad (2)$$

From the equations (1) and (2), we can form an equation

$$A_1x + B_1y + C_1 + k(A_2x + B_2y + C_2) = 0 \quad (3)$$

where k is an arbitrary constant called parameter. For any value of k, the equation (3) is of first degree in x and y. Hence it represents a family of lines. A particular member of this family can be obtained for some value of k. This value of k may be obtained from other conditions.

Example 20 Find the equation of line parallel to the y-axis and drawn through the point of intersection of $x - 7y + 5 = 0$ and $3x + y - 7 = 0$

Soluion The equation of any line through the point of intersection of the given lines is of the form

$$x - 7y + 5 + k(3x + y - 7) = 0$$
$$i.e.,\ (1+3k)x + (k-7)y + 5 - 7k = 0 \qquad (1)$$

If this line is parallel to y-axis, then the coefficient of y should be zero, i.e., $k - 7 = 0$ which gives $k = 7$.

Substituting this value of k in the equation (1), we get

$22x - 44 = 0$, i.e., $x - 2 = 0$, which is the required equation.

EXERCISE 10.4

1. Find the equation of the line through the intersection of lines $3x + 4y = 7$ and $x - y + 2 = 0$ and whose slope is 5.

(Ans. $35x - 7y + 18 = 0$)

2. Find the equation of the line through the intersection of lines $x + 2y - 3 = 0$ and $4x - y + 7 = 0$ and which is parallel to $5x + 4y - 20 = 0$

(Ans. $15x + 12y - 7 = 0$)

3. Find the equation of the line through the intersection of the lines $2x + 3y - 4 = 0$ and $x - 5y = 7$ that has its x-intercept equal to -4.

(Ans. $10x + 93y + 40 = 0$.)

4. Find the equation of the line through the intersection of $5x - 3y = 1$ and $2x + 3y - 23 = 0$ and perpendicular to the line $5x - 3y - 1 = 0$.

(Ans. $63x + 105y - 781 = 0$.)

10.7. Shifting of origin

An equation corresponding to a set of points with reference to a system of coordinate axes may be simplified by taking the set of points in some other suitable coordinate system such that all geometric properties remain unchanged. One such transformation is that in which the new axes are transformed parallel to the original axes and origin is shifted to a new point. A transformation of this kind is called a *translation of axes*.

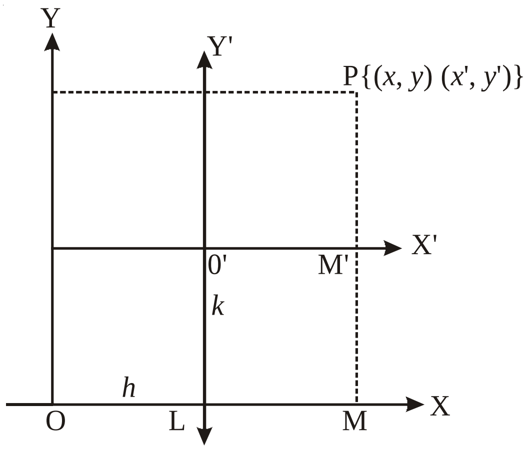


Fig. 10.21

The coordinates of each point of the plane are changed under a translation of axes. By knowing the relationship between the old coordinates and the new coordinates of points, we can study the analytical problem in terms of new system of coordinate axes.

To see how the coordinates of a point of the plane changed under a translation of axes, let us take a point P (x, y) referred to the axes OX and OY. Let O′X′ and O′Y′ be new axes parallel to OX and OY respectively, where O′ is the new origin. Let (h, k) be the coordinates of O′ referred to the old axes, i.e., OL = h and LO′ = k. Also, OM = x and MP = y (see Fig.10.21)

Let $O'M' = x'$ and $M'P = y'$ be respectively, the abscissa and ordinates of a point P referred to the new axes $O'X'$ and $O'Y'$. From Fig.10.21, it is easily seen that

OM = OL + LM, i.e., $x = h + x'$

and MP = MM′ + M′ P, i.e., $y = k + y'$

Hence, $x = x' + h$, $y = y' + k$

These formulae give the relations between the old and new coordinates.

Example 21 Find the new coordinates of point (3, – 4) if the origin is shifted to (1, 2) by a translation.

Solution The coordinates of the new origin are $h = 1$, $k = 2$, and the original coordinates are given to be $x = 3$, $y = -4$.

The transformation relation between the old coordinates (x, y) and the new coordinates (x', y') are given by

$x = x' + h$ i.e., $x' = x - h$

and $y = y' + k$ i.e., $y' = y - k$

Substituting the values, we have

$x' = 3 - 1 = 2$ and $y' = -4 - 2 = -6$

Hence, the coordinates of the point (3, – 4) in the new system are (2, – 6).

Example 22 Find the transformed equation of the straight line $2x - 3y + 5 = 0$, when the origin is shifted to the point (3, –1) after translation of axes.

Solution Let coordinates of a point P changes from (x, y) to (x', y') in new coordinate axes whose origin has the coordinates $h = 3$, $k = -1$. Therefore, we can write the transformation formulae as $x = x' + 3$ and $y = y' - 1$. Substituting, these values in the given equation of the straight line, we get

$$2(x' + 3) - 3(y' - 1) + 5 = 0$$

or

$$2x' - 3y' + 14 = 0$$

Therefore, the equation of the straight line in new system is $2x - 3y + 14 = 0$

EXERCISE 10.5

1. Find the new coordinates of the points in each of the following cases if the origin is shifted to the point (–3, –2) by a translation of axes.

   (i) (1, 1) (Ans (4, 3)) (ii) (0, 1) (Ans. (3, 3)) (iii) (5, 0) (Ans. (8, 2))

   (iv) (–1, –2) (Ans. (2, 0)) (v) (3, –5) (Ans. (6, –3))

2. Find what the following equations become when the origin is shifted to the point (1, 1)

   (i) $x^2 + xy - 3y^2 - y + 2 = 0$ (Ans. $x^2 - 3y^2 + xy + 3x - 6y + 1 = 0$)

   (ii) $xy - y^2 - x + y = 0$ (Ans. $xy - y^2 = 0$)

   (iii) $xy - x - y + 1 = 0$ (Ans. $xy = 0$)

CHAPTER 13

13.5. Limits involving exponential and logarithmic functions

Before discussing evaluation of limits of the expressions involving exponential and logarithmic functions, we introduce these two functions stating their domain, range and also sketch their graphs roughly.

Leonhard Euler (1707AD – 1783AD), the great Swiss mathematician introduced the number e whose value lies between 2 and 3. This number is useful in defining exponential function and is defined as $f(x) = e^x$, $x \in \mathbf{R}$. Its domain is $\mathbf{R}$, range is the set of positive real numbers. The graph of exponential function, i.e., $y = e^x$ is as given in Fig.13.11.

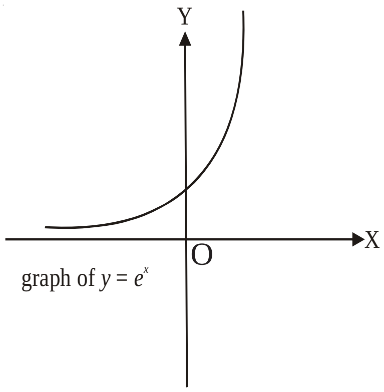


Fig.13.11

Similarly, the logarithmic function expressed as $\log_e : \mathbf{R}^+ \to \mathbf{R}$ is given by $\log_e x = y$, if and only if $e^y = x$. Its domain is $\mathbf{R}^+$ which is the set of all positive real numbers and range is $\mathbf{R}$. The graph of logarithmic function $y = \log_e x$ is shown in Fig.13.12.

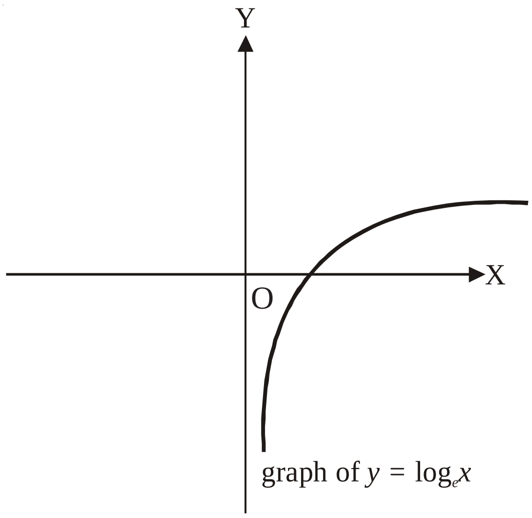


Fig. 13.12

In order to prove the result $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$, we make use of an inequality involving the expression $\frac{e^x - 1}{x}$ which runs as follows:

$$\frac{1}{1+|x|} \le \frac{e^x - 1}{x} \le 1 + (e-2)|x| \text{ holds for all } x \text{ in } [-1, 1] \sim \{0\}.$$

Theorem 6 Prove that $\lim_{x\to 0}\frac{e^x-1}{x}=1$

Proof Using above inequality, we get

$$\frac{1}{1+|x|}\le \frac{e^x-1}{x} \le 1+|x|(e-2),\ x\in [-1,1]\sim\{0\}$$

Also $$\lim_{x\to 0}\frac{1}{1+|x|}=\frac{1}{1+\lim_{x\to 0}|x|}=\frac{1}{1+0}=1$$

and $$\lim_{x\to 0}\left[1+(e-2)|x|\right]=1+(e-2)\lim_{x\to 0}|x|=1+(e-2)0=1$$

Therefore, by Sandwich theorem, we get

$$\lim_{x\to 0}\frac{e^x-1}{x}=1$$

Theorem 7 Prove that $\lim_{x\to 0}\frac{\log_e(1+x)}{x}=1$

Proof Let $\frac{\log_e(1+x)}{x}=y$. Then

$$\log_e(1+x)=xy$$

$$\Rightarrow 1+x=e^{xy}$$

$$\Rightarrow \frac{e^{xy}-1}{x}=1$$

or $$\frac{e^{xy}-1}{xy}.y=1$$

$$\Rightarrow \lim_{xy\to 0}\frac{e^{xy}-1}{xy}\lim_{x\to 0}y=1 (\text{since } x\to 0 \text{ gives } xy\to 0)$$

$$\Rightarrow \lim_{x\to 0}y=1\left(\text{as} \lim_{xy\to 0}\frac{e^{xy}-1}{xy}=1\right)$$

$$\Rightarrow \lim_{x\to 0}\frac{\log_e(1+x)}{x}=1$$

Example 5 Compute $\lim_{x\to 0}\frac{e^{3x}-1}{x}$

Solution We have

$$\lim_{x\to 0}\frac{e^{3x}-1}{x}=\lim_{3x\to 0}\frac{e^{3x}-1}{3x}\cdot 3$$

$$=3\left(\lim_{y\to 0}\frac{e^y-1}{y}\right),\quad \text{where } y=3x$$

$$=3.1=3$$

Example 6 Compute $\lim_{x\to 0}\frac{e^x - \sin x - 1}{x}$

Solution We have $\lim_{x\to 0}\frac{e^x - \sin x - 1}{x} = \lim_{x\to 0}\left[\frac{e^x - 1}{x} - \frac{\sin x}{x}\right]$

$$= \lim_{x\to 0}\frac{e^x - 1}{x} - \lim_{x\to 0}\frac{\sin x}{x} = 1 - 1 = 0$$

Example 7 Evaluate $\lim_{x\to 1}\frac{\log_e x}{x-1}$

Solution Put $x = 1 + h$, then as $x \to 1 \Rightarrow h \to 0$. Therefore,

$$\lim_{x\to 1}\frac{\log_e x}{x-1} = \lim_{h\to 0}\frac{\log_e (1+h)}{h} = 1\left(\text{since } \lim_{x\to 0}\frac{\log_e (1+x)}{x} = 1\right)$$

EXERCISE 13.2

Evaluate the following limits, if exist

1. $\lim_{x\to 0}\frac{e^{4x} - 1}{x}$ (Ans. 4)

2. $\lim_{x\to 0}\frac{e^{2+x} - e^2}{x}$ (Ans. e^{2})

3. $\lim_{x\to 5}\frac{e^x - e^5}{x-5}$ (Ans. e^5)

4. $\lim_{x\to 0}\frac{e^{\sin x} - 1}{x}$ (Ans. 1)

5. $\lim_{x\to 3}\frac{e^x - e^3}{x-3}$ (Ans. e^3)

6. $\lim_{x\to 0}\frac{x(e^x - 1)}{1 - \cos x}$ (Ans. 2)

7. $\lim_{x\to 0}\frac{\log_e (1+2x)}{x}$ (Ans. 2)

8. $\lim_{x\to 0}\frac{\log (1+x^3)}{\sin^3 x}$ (Ans. 1)

Annexure II

CLASS XII

CHAPTER 5

Theorem 5 (To be inserted on page 173 under the heading theorem 5)

(i) Derivative of Exponential Function $f(x) = e^x$.

If $f(x) = e^x$, then

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x}$$

$$= e^x \cdot \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x}$$

$$= e^x \cdot 1 \text{ [since } \lim_{h \to 0} \frac{e^h - 1}{h} = 1]$$

Thus, $\frac{d}{dx}(e^x) = e^x$.

(ii) Derivative of logarithmic function $f(x) = \log_e x$.

If $f(x) = \log_e x$, then

$$f'(x) = \lim_{\Delta x \to 0} \frac{\log_e(x+\Delta x) - \log_e x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\log_e\left(1 + \frac{\Delta x}{x}\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{x} \frac{\log_e\left(1 + \frac{\Delta x}{x}\right)}{\frac{\Delta x}{x}}$$

$$= \frac{1}{x} \text{ [since } \lim_{h \to 0} \frac{\log_e(1+h)}{h} = 1]$$

Thus, $\frac{d}{dx}\log_e x = \frac{1}{x}$

7.6.3. $\int (\boldsymbol{px+q})\sqrt{\boldsymbol{ax^2+bx+c}}\, \boldsymbol{dx}$.

We choose constants A and B such that

$$px + q = \mathrm{A}\left[\frac{d}{dx}(ax^2+bx+c)\right]+\mathrm{B}$$

$$= \mathrm{A}(2ax + b) + \mathrm{B}$$

Comparing the coefficients of x and the constant terms on both sides, we get

$$2a\mathrm{A} = p \text{ and } \mathrm{A}b + \mathrm{B} = q$$

Solving these equations, the values of A and B are obtained. Thus, the integral reduces to

$$\mathrm{A}\int(2ax+b)\sqrt{ax^2+bx+c}\,dx+\mathrm{B}\int\sqrt{ax^2+bx+c}\,dx$$

$$= \mathrm{AI_1}+\mathrm{BI_2}$$

where $\quad \mathrm{I_1} = \int(2ax+b)\sqrt{ax^2+bx+c}\,dx$

Put $ax^2 + bx + c = t$, then $(2ax + b)dx = dt$

So $\quad \mathrm{I_1} = \dfrac{2}{3}(ax^2+bx+c)^{\frac{3}{2}}+\mathrm{C_1}$

Similarly, $\quad \mathrm{I_2} = \int\sqrt{ax^2+bx+c}\,dx$

is found, using the integral formula discussed in [7.6.2, Page 328 of the textbook].

Thus $\int(px+q)\sqrt{ax^2+bx+c}\,dx$ is finally worked out.

Example 25 Find $\int x\sqrt{1+x-x^2}\,dx$

Solution Following the procedure as indicated above, we write

$$x = \mathrm{A}\left[\frac{d}{dx}\left(1+x-x^2\right)\right]+\mathrm{B}$$

$$= \mathrm{A}\,(1-2x)+\mathrm{B}$$

Equating the coefficients of x and constant terms on both sides,

We get $-2\mathrm{A} = 1$ and $\mathrm{A} + \mathrm{B} = 0$

Solving these equations, we get $\mathrm{A} = -\dfrac{1}{2}$ and $\mathrm{B} = \dfrac{1}{2}$. Thus the integral reduces to

$$\int x\sqrt{1+x-x^2}\,dx = -\frac{1}{2}\int (1-2x)\sqrt{1+x-x^2}\,dx + \frac{1}{2}\int \sqrt{1+x-x^2}\,dx$$

$$= -\frac{1}{2}\mathrm{I}_1 + \frac{1}{2}\mathrm{I}_2 \qquad (1)$$

Consider $\mathrm{I}_1 = \int (1-2x)\sqrt{1+x-x^2}\,dx$

Put $1 + x - x^2 = t$, then $(1 - 2x)dx = dt$

Thus $\mathrm{I}_1 = \int (1-2x)\sqrt{1+x-x^2}\,dx = \int t^{\frac{1}{2}}dt = \frac{2}{3}t^{\frac{3}{2}} + \mathrm{C}_1$

$= \frac{2}{3}\left(1+x-x^2\right)^{\frac{3}{2}} + \mathrm{C}_1$, where C_1 is some constant.

Further, consider $\mathrm{I}_2 = \int \sqrt{1+x-x^2}\,dx = \int \sqrt{\frac{5}{4} - \left(x-\frac{1}{2}\right)^2}\,dx$

Put $x - \frac{1}{2} = t.$ Then $dx = dt$

Therefore, $\mathrm{I}_2 = \int \sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - t^2}\,dt$

$$= \frac{1}{2}t\sqrt{\frac{5}{4}-t^2} + \frac{1}{2}\cdot\frac{5}{4}\sin^{-1}\frac{2t}{\sqrt{5}} + \mathrm{C}_2$$

$$= \frac{1}{2}\frac{(2x-1)}{2}\sqrt{\frac{5}{4}-(x-\frac{1}{2})^2} + \frac{5}{8}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + C_2$$

$= \frac{1}{4}(2x-1)\sqrt{1+x-x^2} + \frac{5}{8}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + C_2$, where C_2 is some constant.

Putting values of I_1 and I_2 in (1), we get

$$\int x\sqrt{1+x} - x^2 dx = -\frac{1}{3}(1+x-x^2)^{\frac{3}{2}} + \frac{1}{8}(2x-1)\sqrt{1+x-x^2}$$

$$+\frac{5}{16}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + C,$$

where $\mathrm{C} = -\frac{\mathrm{C}_1+\mathrm{C}_2}{2}$ is another arbitrary constant.

Insert the following exercises at the end of EXERCISE 7.7 as follows:

12. $x\sqrt{x+x^2}$ 13. $(x+1)\sqrt{2x^2+3}$ 14. $(x+3)\sqrt{3-4x-x^2}$

Answers

12. $$\frac{1}{3}(x^2+x)^{\frac{3}{2}}-\frac{(2x+1)\sqrt{x^2+x}}{8}+\frac{1}{16}\log\left|x+\frac{1}{2}+\sqrt{x^2+x}\right|+C$$

13. $$\frac{1}{6}(2x^2+3)^{\frac{3}{2}}+\frac{x}{2}\sqrt{2x^2+3}+\frac{3\sqrt{2}}{4}\log\left|x+\sqrt{x^2+\frac{3}{2}}\right|+C$$

14. $$-\frac{1}{3}(3-4x-x^2)^{\frac{3}{2}}+\frac{7}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{7}}\right)+\frac{(x+2)\sqrt{3-4x-x^2}}{2}+C$$

CHAPTER 10

10.7 Scalar triple product Let $\vec{a},\vec{b}$ and $\vec{c}$ be any three vectors. The scalar product of $\vec{a}$ and $(\vec{b}\times\vec{c})$, i.e., $\vec{a}\cdot(\vec{b}\times\vec{c})$ is called the scalar triple product of $\vec{a},\vec{b}$ and $\vec{c}$ in this order and is denoted by $[\vec{a},\vec{b},\vec{c}]$ (or $[\vec{a}\,\vec{b}\,\vec{c}]$). We thus have

$$[\vec{a},\vec{b},\vec{c}]=\vec{a}\cdot(\vec{b}\times\vec{c})$$

Observations

1. Since $(\vec{b}\times\vec{c})$ is a vector, $\vec{a}\cdot(\vec{b}\times\vec{c})$ is a scalar quantity, i.e. $[\vec{a},\vec{b},\vec{c}]$ is a scalar quantity.

2. Geometrically, the magnitude of the scalar triple product is the volume of a parallelopiped formed by adjacent sides given by the three vectors $\vec{a},\vec{b}$ and $\vec{c}$ (Fig. 10.28). Indeed, the area of the parallelogram forming the base of the parallelopiped is $\left|\vec{b}\times\vec{c}\right|$. The height is the projection of $\vec{a}$ along the normal to the plane containing $\vec{b}$ and $\vec{c}$ which is the magnitude of the component of $\vec{a}$ in the direction of $\vec{b}\times\vec{c}$ i.e., $\dfrac{\left|\vec{a}.(\vec{b}\times\vec{c})\right|}{\left|(\vec{b}\times\vec{c})\right|}$. So the required volume of the parallelopiped is $\dfrac{\left|\vec{a}.(\vec{b}\times\vec{c})\right|}{\left|(\vec{b}\times\vec{c})\right|}\,|\vec{b}\times\vec{c}|=\left|\vec{a}.(\vec{b}\times\vec{c})\right|$,

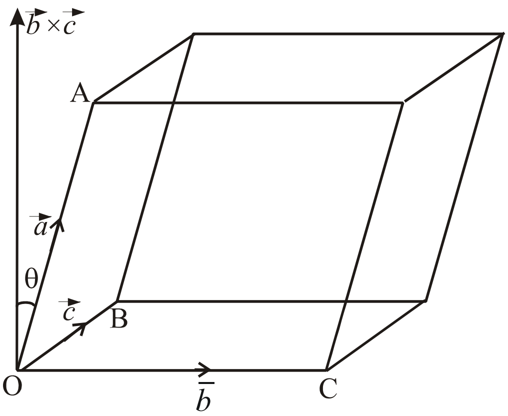


Fig. 10.28

3. If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$, then

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= (b_2c_3 - b_3c_2)\ \hat{i} + (b_3c_1 - b_1c_3)\ \hat{j}\ + (b_1c_2 - b_2c_1)\ \hat{k}$$

and so

$$\vec{a}.(\vec{b} \times \vec{c}) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3\,(b_1c_2 - b_2c_1)$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

4. If $\vec{a}, \vec{b}$ and $\vec{c}$ be any three vectors, then

$$[\vec{a}, \vec{b}\ ,\ \vec{c}\ ] = [\vec{b}, \vec{c}, \vec{a}\ ] = [\vec{c}, \vec{a}, \vec{b}\ ]$$

(cyclic permutation of three vectors does not change the value of the scalar triple product).

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k},\ \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j}_3\hat{k}.$ Then, just by observation above, we have

$$[\vec{a}, \vec{b}, \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1\ (b_2c_3 - b_3c_2) + a_2\ (b_3c_1 - b_1c_3) + a_3\ (b_1c_2 - b_2c_1)$$
$$= b_1\ (a_3c_2 - a_2c_3) + b_2\ (a_1c_3 - a_3c_1) + b_3\ (a_2c_1 - a_1c_2)$$

$$= \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= [\vec{b}, \vec{c}, \vec{a}\ ]$$

Similarly, the reader may verify that

$$= [\vec{a}, \vec{b}\ ,\ \vec{c}\ ] = [\vec{c}, \vec{a}, \vec{b}\ ]$$

Hence $\quad [\vec{a}, \vec{b}\ ,\ \vec{c}\ ] = [\vec{b}, \vec{c}, \vec{a}\ ] = [\vec{c}, \vec{a}, \vec{b}\ ]$

5. In scalar triple product $\vec{a}.(\vec{b} \times \vec{c})$, the dot and cross can be interchanged. Indeed,

$$\vec{a}.(\vec{b} \times \vec{c}) = [\vec{a}, \vec{b}, \vec{c}\ ] = [\vec{b}, \vec{c}, \vec{a}\ ] = [\vec{c}, \vec{a}, \vec{b}\ ] = \vec{c}.(\vec{a} \times \vec{b}) = (\vec{a} \times \vec{b}).\vec{c}$$

6. $= [\vec{a}, \vec{b}, \vec{c}\ ] = -[\vec{a}, \vec{c}, \vec{b}\ ]$. Indeed

$= [\vec{a}, \vec{b}, \vec{c}\,] = \vec{a}.(\vec{b} \times \vec{c})$

$= \vec{a}.(-\vec{c} \times \vec{b})$

$= -(\vec{a}.(\vec{c} \times \vec{b}))$

$= -\left[\vec{a}, \vec{c}, \vec{b}\right]$

7. $[\vec{a}, \vec{a}, \vec{b}] = 0.$ Indeed

$[\vec{a}, \vec{a}, \vec{b}] = [\vec{a}, \vec{b}, \vec{a},]$

$= [\vec{b}, \vec{a}, \vec{a}]$

$= \vec{b}.(\vec{a} \times \vec{a})$

$= \vec{b}.\vec{0} = 0.$ (as $\vec{a} \times \vec{a} = \vec{0}$)

Note: The result in 7 above is true irrespective of the position of two equal vectors.

10.7.1 Coplanarity of three vectors

Theorem 1 Three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar if and only if $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$.

Proof : Suppose first that the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar.

If $\vec{b}$ and $\vec{c}$ are parallel vectors, then, $\vec{b} \times \vec{c} = \vec{0}$ and so $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$.

If $\vec{b}$ and $\vec{c}$ are not parallel then, since $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar, $\vec{b} \times \vec{c}$ is perpendicular to $\vec{a}$.

So $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$.

Conversely, suppose that $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$. If $\vec{a}$ and $\vec{b} \times \vec{c}$ are both non-zero, then we conclude that $\vec{a}$ and $\vec{b} \times \vec{c}$ are perpendicular vectors. But $\vec{b} \times \vec{c}$ is perpendicular to both $\vec{b}$ and $\vec{c}$. Therefore $\vec{a}$ and $\vec{b}$ and $\vec{c}$ must lie in the plane, i.e. they are coplanar. If $\vec{a} = 0$, then $\vec{a}$ is coplanar with any two vectors, in particular with $\vec{b}$ and $\vec{c}$. If $(\vec{b} \times \vec{c}) = 0$, then $\vec{b}$ and $\vec{c}$ are parallel vectors and so, $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar since any two vectors always lie in a plane determined by them and a vector which is parallel to any one of it also lies in that plane.

Note: Coplanarity of four points can be discussed using coplanarity of three vectors. Indeed, the four points A, B, C and D are coplanar if the vectors $\overrightarrow{AB}, \overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar.

Example 26: Find $\vec{a}.(\vec{b} \times \vec{c})$, *if* $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2j + k$ and $\vec{c} = 3\hat{i} + \hat{j} + 2\hat{k}$.

Solution : We have $\vec{a}.(\vec{b} \times \vec{c}) = \begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix} = -10.$

Example 27: Show that the vectors $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b} = -2\hat{i} + 3j - 4\hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} + 5\hat{k}$ are coplanar.

Solution : We have $\vec{a}.(\vec{b} \times \vec{c}) = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 2 & -4 \\ 1 & -3 & 5 \end{vmatrix} = 0.$

Hence, in view of Theorem 1, $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar vectors.

Example 28: Find λ if the vectors $\vec{a} = \hat{i} + 3\hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} - \hat{k}$ and $\vec{c} = \lambda\hat{i} + 7\hat{j} + 3\hat{k}$ are coplanar.

Solution : Since $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar vectors, we have $\left[\vec{a}, \vec{b}, \vec{c}\right] = 0$, i.e.,

$$\begin{vmatrix} 1 & 3 & 1 \\ 2 & -1 & -1 \\ \lambda & 7 & 3 \end{vmatrix} = 0.$$

$\Rightarrow \quad 1\,(-3 + 7) - 3\,(6 + \lambda) + 1\,(14 + \lambda) = 0$

$\Rightarrow \quad \lambda = 0.$

Example 29: Show that the four points A, B, C and D with position vectors $4\hat{i} + 5\hat{j} + \hat{k}, -(\hat{j} + \hat{k}), 3\hat{i} + 9\hat{j} + 4\hat{k}$ and $4(-\hat{i} + \hat{j} + \hat{k})$, respectively are coplanar.

Solution : We know that the four points A, B, C and D are coplanar if the three vectors $\overrightarrow{AB}, \overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar, i.e., if

$$\left[\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}\right] = 0$$

Now $\quad \overrightarrow{AB} = -(\hat{j} + \hat{k}) - (4\hat{i} + 5\hat{j} + \hat{k}) = -4\hat{i} - 6\hat{j} - 2\hat{k})$

$\overrightarrow{AC} = (3\hat{i} + 9\hat{j} + 4\hat{k}) - (4\hat{i} + 5\hat{j} + \hat{k}) = -\hat{i} + 4\hat{j} + 3\hat{k}$

and $\quad \overrightarrow{AD} = 4(-\hat{i} + \hat{j} + \hat{k}) - (4\hat{i} + 5\hat{j} + \hat{k}) = -8\hat{i} - \hat{j} + 3\hat{k}$

Thus $\quad \left[\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}\right] = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix} = 0.$

Hence A, B, C and D are coplanar.

Example 30 : Prove that $\left[\vec{a} + \vec{b}, \vec{b} + \vec{c}, \vec{c} + \vec{a}\right] = 2\left[\vec{a}, \vec{b}, \vec{c}\right]$.

Solution : We have

$$\left[\vec{a} + \vec{b}, \vec{b} + \vec{c}, \vec{c} + \vec{a}\right] = (\vec{a} + \vec{b}).((\vec{b} + \vec{c}) \times (\vec{c} + \vec{a}))$$

$$= (\vec{a} + \vec{b}).(\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{c} + \vec{c} \times \vec{a})$$

$$= (\vec{a} + \vec{b}).(\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a}) \qquad (\text{as } \vec{c} \times \vec{c} = \vec{0})$$

$$= \vec{a}.(\vec{b} \times \vec{c}) + \vec{a}.(\vec{b} \times \vec{a}) + \vec{a}.(\vec{c} \times \vec{a}) + \vec{b}.(\vec{b} \times \vec{c}) + \vec{b}.(\vec{b} \times \vec{a}) + \vec{b}.(\vec{c} \times \vec{a})$$

$$= \left[\vec{a}, \vec{b}, \vec{c}\right] + \left[\vec{a}, \vec{b}, \vec{a}\right] + \left[\vec{a}, \vec{c}, \vec{a}\right] + \left[\vec{b}, \vec{b}, \vec{c}\right] + \left[\vec{b}, \vec{b}, \vec{a}\right] + \left[\vec{b}, \vec{c}, \vec{a}\right]$$

$$= 2\left[\vec{a}, \vec{b}, \vec{c}\right] \qquad \text{(Why ?)}$$

Example 31 : Prove that $\left[\vec{a}, \vec{b}, \vec{c} + \vec{d}\right] = \left[\vec{a}, \vec{b}, \vec{c}\right] + [\vec{a}, \vec{b}, \vec{d}]$

Solution We have

$$\left[\vec{a}, \vec{b}, \vec{c} + \vec{d}\right] = \vec{a} . (\vec{b} \times (\vec{c} + \vec{d}))$$

$$= \vec{a} . (\vec{b} \times \vec{c} + \vec{b} \times \vec{d})$$

$$= \vec{a} . (\vec{b} \times \vec{c}) + \vec{a} . (\vec{b} \times \vec{d})$$

$$= \left[\vec{a}, \vec{b}, \vec{c}\right] + \left[\vec{a}, \vec{b}, \vec{d}\right]$$

EXERCISE 10.5

1. Find $\left[\vec{a}\, \vec{b}\, \vec{c}\right]$ if $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}, \vec{b} = 2\hat{i} - 3\hat{j} + \hat{k}$ and $c = 3i + j - 2\hat{k}$

   (Answer : 24)

2. Show that the vectors $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}, \vec{b} = -2\hat{i} + 3\hat{j} - 4\hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} + 5\hat{k}$ are coplanar.

3. Find λ if the vectors $\hat{i} - \hat{j} + \hat{k}, 3\hat{i} + \hat{j} + 2\hat{k}$ and $\hat{i} + \lambda\hat{j} - 3\hat{k}$ are coplanar. (Answer : $\lambda = 15$)

4. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}, \vec{b} = \hat{i}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ Then

   (a) If $c_1 = 1$ and $c_2 = 2$, find c_3 which makes $\vec{a}, \vec{b}$ and $\vec{c}$ coplanar (Answer : $c_3 = 2$)

   (b) If $c_2 = -1$ and $c_3 = 1$, show that no value of c_1 can makes $\vec{a}, \vec{b}$ and $\vec{c}$ coplanar.

5. Show that the four points with position vectors

   $4\hat{i} + 8\hat{j} + 12\hat{k}, 2\hat{i} + 4\hat{j} + 6\hat{k}, 3\hat{i} + 5\hat{j} + 4\hat{k}$ and $5\hat{i} + 8\hat{j} + 5\hat{k}$ are coplanar.

6. Find x such that the four points A (3, 2, 1) B (4, x, 5), C (4, 2, –2) and D (6, 5, –1) are coplanar. (Answer $x = 5$)

7. Show that the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ coplanar if $\vec{a} + \vec{b}$, $\vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ are coplanar.

MATHEMATICS